

Lax-Phillips Theory and Quantum Evolution

L.P. Horwitz^{§†}, E. Eisenberg[§] and Y. Strauss[†]

[§] Department of Physics, Bar-Ilan University
Ramat-Gan 52900, Israel

[†] School of Physics and Astronomy,
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University, Ramat-Aviv, Israel

Abstract: The scattering theory of Lax and Phillips, designed primarily for hyperbolic systems, such as electromagnetic or acoustic waves, is described. This theory provides a realization of the theorem of Foias and Nagy; there is a subspace of the Hilbert space in which the unitary evolution of the system, restricted to this subspace, is realized as a semigroup. The embedding of the quantum theory into this structure, carried out by Flesia and Piron, is reviewed. We show how the density matrix for an effectively pure state can evolve to an effectively mixed state (decoherence) in this framework. Necessary conditions are given for the realization of the relation between the spectrum of the generator of the semigroup and the singularities of the S -matrix (in energy representation). It is shown that these conditions may be met in the Liouville space formulation of quantum evolution, and in the Hilbert space of relativistic quantum theory.

1. INTRODUCTION

The unstable quantum system is an important example of irreversible phenomena in nature. Such systems, ranging from excited atomic states to short-lived elementary particles, are characterized by what is generally observed to be an irreversible evolution. These phenomena raise the question of explanation of such processes from first principles. Moreover, since most of the decay processes are observed experimentally to obey an exponential decay law, one expects this behavior to follow from very general assumptions.

Irreversible evolution in the quantum theory has been described by the addition of non-Hermitian terms to the Hamiltonian, such that it has complex eigenvalues, and the induced evolution is non-unitary. Structures of this type were originally introduced by Gamow [1] who studied the effect of assigning complex eigenvalues to the energy spectrum, and hence introduced a kind of generalized eigenvector. Wu and Yang [2] parameterized the K -meson decay in this way. In this method, the non-Hermitian terms in the Hamiltonian are introduced phenomenologically, and may only indirectly be associated with some known interaction in a more fundamental Hamiltonian.

Weisskopf and Wigner [3], in a well known paper in 1930, introduced an alternative approach to the decay problem. According to their approach, the evolution takes place in a Hilbert space which is a direct sum of two subspaces: the subspace of the decaying states and that of decay products. These two subspaces are stable under the “free” evolution induced by H_0 , but are combined linearly under the full evolution induced by

$H = H_0 + V$. In this Hilbert space, the evolution is unitary, and hence its generator, i.e., the Hamiltonian, is self-adjoint. The decay is described as the probability flow from the subspace of the decaying states to its complement, the subspace of the decay products. They studied perturbatively, for the single-channel case, what has become known as the *survival amplitude*

$$A(t) = (\psi, e^{-iHt}\psi), \quad (1.1)$$

which is the probability amplitude for the system to remain in the discrete state until time t . In the following we will describe this approach, and pose critical problems, motivating the development of a more general theory.

Let us denote the projection operators on these two subspaces as P and \bar{P} , such that $P + \bar{P} = 1$. For the decay problem, the basic quantity is the *reduced motion*

$$U'(t) = PU(t)P. \quad (1.2)$$

where $U(t) = e^{-iHt}$, which governs the time evolution of the subspace $P\mathcal{H}$ of the unstable states. From this one can derive the decay law of the unstable states. If $\{\phi_i\}$ is an orthonormal basis of $P\mathcal{H}$, the probability that an unstable state ϕ , which exists at time $t = 0$, is in the subspace $P\mathcal{H}$ of unstable states at time t is given by

$$p(t) = \sum_i |(\phi_i, U(t)\phi)|^2 = \text{Tr}(U'(t)^\dagger U'(t)P_\phi), \quad (1.3)$$

where $P_\phi = |\phi\rangle\langle\phi|$.

The total evolution operator $U(t) = e^{-iHt}$, and the resolvent $R(z) = (z - H)^{-1}$ are related to each other by the (inverse) Laplace transform

$$U(t) = \frac{1}{2\pi i} \oint R(z)e^{-izt}dz, \quad (1.4)$$

where the integration contour is around the spectrum of H . If we project this operator into the subspace $P\mathcal{H}$, we can obtain a similar relation which expresses the reduced motion $U'(t)$ in terms of the *reduced resolvent* $R'(z) = PR(z)P$:

$$U'(t) = \frac{1}{2\pi i} \oint R'(z)e^{-izt}dz. \quad (1.5)$$

By differentiating Eq. (1.3) and setting $t = 0$, one sees that the initial decay rate is necessarily zero (providing that the Hamiltonian is defined on the initial state); in fact, it is easy to show that the change in $p(t)$ is $O(t^2)$. The intermediate and long time behavior follow most simply by an examination of the relation (1.5). Deforming the contour of integration which runs below the real positive spectrum of H to the negative imaginary axis, where its contribution for large times is small, the remaining contribution of the contour running above the spectrum of H can be estimated by bringing this contour continuously through the cut. When resonances exist, this contour will pass through simple poles on the way to the negative imaginary axis in the second sheet (as can be explicitly

demonstrated [4] in the Lee-Friedrichs model [5]). The residues of the poles may dominate the time dependence for intermediate times, and give the approximate (due to the presence of residual contributions from the integrals along the negative imaginary axis) exponential decay behavior. For very long times, the pole contributions disappear, and the remaining integration around the branch cut results in an inverse power law asymptotic behavior [6].

It is not difficult to see that an irreversible evolution must be described by a semigroup [7] (for the reversible case this is a group induced by a unitary transformation), where we define a semigroup as follows:

Let $\{Z(t)\}$ be a family (over t) of operators on a Hilbert space; then $Z(t)$ is an element of a semigroup if

$$Z(t_1)Z(t_2) = Z(t_1 + t_2), \quad t_1, t_2 \geq 0 \quad (1.6)$$

The semigroup is said to be *strongly contractive* if $\|Z(t)\| \rightarrow 0$, for $t \rightarrow \infty$, where $\|A\|$ is the operator norm of A . On the other hand, it can be shown that the reduced motion, as described above, cannot generate a semigroup [8].

There is, furthermore, another, perhaps more fundamental problem associated with the general method of Wigner and Weisskopf; this is that the expression (1.1) for the survival amplitude implicitly assumes the existence of a linear superposition (we restrict our discussion here to the one-channel case)

$$e^{-iHt}\psi = A(t)\psi + \chi(t), \quad (1.7)$$

where $\chi(t)$ represents the decayed system and $(\psi, \chi(t)) = 0$. In general this linear superposition does not correspond to any physical situation in our experience; a short-lived particle, for example, is seen as either the particle before the decay, or the decay products at a certain time, which can not be predicted. This linear superposition does not correspond to the object that we see experimentally in such a process.

In the framework of the theory of Weisskopf and Wigner, techniques have been developed which are capable of displaying the exact semigroup behavior of an unstable system [9]. As we have remarked above, the Lee-Friedrichs model [5] provides a simple but useful example of an unstable system for which the evolution equations are completely soluble [4]. The eigenvalue equation $Hf(z) = zf(z)$, for $f \in \mathcal{H}$ has a formal solution which does not, however, satisfy the equation. The condition to satisfy the equation coincides with the condition for a pole in the resolvent $R'(z)$, and can only be satisfied in the second Riemann sheet. The scalar product of the eigenvalue equation with a vector for which the unperturbed energy representation is analytic in a domain containing the second sheet pole can be continued to the second sheet, and at the pole position, the equation is identically satisfied. Since the “eigenvector” obtained in this way is in the space dual to a set of vectors with this restrictive analyticity requirement, corresponding to a subspace of \mathcal{H} , it is an element in the large space of a Gel’fand triple (rigged Hilbert space). The definition of this vector depends on the domain of analyticity chosen, and its physical interpretation is not clear, except for the fact that its (extended) unitary evolution is that of an exact exponential decay.

The problems discussed above are essentially related to the attempt to describe an unstable system in a framework more suitable to the description of reversible phenomena.

In what follows we will show another approach to irreversible phenomena which attempts to solve these difficulties.

2. LAX-PHILLIPS THEORY AND THE EXACT SEMIGROUP

The characterization of a system undergoing an irreversible process cannot, in principle, be specified at a given instant of time. In fact, the physical quantities describing such processes involve time measurements (that is, measurements of the time at which certain defined phenomena occur). Therefore, the information about the decay which is to be deduced from the state is associated with its distribution in time, an essential property of the system, just as the location or momentum of a quantal particle. The time variable is, from this point of view, an internal degree of freedom of the system, which provides a framework for the description of interactions which can influence the structure of the state [10]. The dynamical evolution of the system involves a change in its internal structure, including its distribution in t along with other observables characterizing the state. This evolution, parameterized by the laboratory time τ (which is not a dynamical variable), is defined on a Hilbert space $\bar{\mathcal{H}}$ with t in its measure space with norm given by (e.g., with Lebesgue measure)

$$\int \|\psi_t^\tau\|^2 dt = \|\psi^\tau\|^2, \quad (2.1)$$

where the norm in the integral is taken as the norm in \mathcal{H}_t , a member of a family of auxiliary Hilbert spaces (all isomorphic), defined for each t .

The theory of Lax and Phillips [11], designed for systems of hyperbolic differential equations describing the scattering of, e.g., electromagnetic or acoustic waves, and the Floquet theory [12] for periodic time dependent quantum mechanical problems are examples of such a structure. Piron [7] has shown that methods of this type are applicable to the general time dependent quantum mechanical problem. Recently, Flesia and Piron [13] have shown that scattering problems in quantum theory can be put in the form of Lax-Phillips theory (Horwitz and Piron [14] have discussed its applicability to the problem of the unstable system) by forming a direct integral of the quantum mechanical Hilbert spaces \mathcal{H}_t over t in order to construct a larger space $\bar{\mathcal{H}}$ which includes t in its measure space.

Lax-Phillips theory [11] assumes the existence of a one-parameter unitary group of evolution on a Hilbert space $\bar{\mathcal{H}}$, and incoming and outgoing subspaces \mathcal{D}_- and \mathcal{D}_+ such that

$$U(\tau)\mathcal{D}_+ \subset \mathcal{D}_+, \text{ for all } \tau > 0$$

$$U(\tau)\mathcal{D}_- \subset \mathcal{D}_-, \text{ for all } \tau < 0$$

$$\begin{aligned} \bigcap_{\tau} U(\tau)\mathcal{D}_{\pm} &= \{0\} \\ \overline{\bigcup_{\tau} U(\tau)\mathcal{D}_{\pm}} &= \bar{\mathcal{H}} \end{aligned} \quad (2.2)$$

where τ is the evolution parameter identified with the laboratory time. It follows from a theorem of Sinai [15] that $\bar{\mathcal{H}}$ can be foliated in such a way that it can be represented as a

family of (auxiliary) Hilbert spaces in the form $L^2(-\infty, +\infty; \mathcal{H}_t)$, over Lebesgue measure in t , and all the \mathcal{H}_t are isomorphic (we therefore sometimes refer to these spaces simply as \mathcal{H}) and determined up to unitary equivalence. The scalar product in $\bar{\mathcal{H}}$ is given by

$$(f, g) = \int_{-\infty}^{\infty} (f_t, g_t)_{\mathcal{H}_t} dt. \quad (2.3)$$

Lax and Phillips show that there are unitary operators W_+^{-1}, W_-^{-1} which map the elements of $\bar{\mathcal{H}}$ into representations, called the outgoing and incoming translation representations, for which the evolution is translation in t . The subspaces $\mathcal{D}_+, \mathcal{D}_-$ correspond to the sets of functions with, in these representations, support in semi-infinite segments of the positive and negative t -axis respectively. They define the S matrix abstractly as the map from the incoming translation representation to the outgoing one, i.e., $S = W_+^{-1} W_-$. This map is defined up to unitary transformations on the auxiliary spaces $\{\mathcal{H}_t\}$, and refers to the equivalence classes for which the incoming and outgoing representations have the property that the evolution is represented by translation.

Lax and Phillips furthermore define the operator

$$\mathcal{Z}(\tau) = P_+ U(\tau) P_- \quad (2.4)$$

on $\bar{\mathcal{H}}$, where P_{\pm} is the projection on the orthogonal complement of \mathcal{D}_{\pm} . This operator vanishes on \mathcal{D}_{\pm} and maps the subspace

$$\mathcal{K} = \bar{\mathcal{H}} \ominus (\mathcal{D}_+ \oplus \mathcal{D}_-), \quad (2.5)$$

into itself. These mappings form a semigroup [11], i.e., for $\tau_1, \tau_2 \geq 0$,

$$\mathcal{Z}(\tau_1) \mathcal{Z}(\tau_2) = \mathcal{Z}(\tau_1 + \tau_2), \quad (2.6)$$

and this semigroup is strongly contractive, i.e., for each $\phi \in \mathcal{K}$ and any ϵ , there exists a τ_{ϕ} such that

$$\|\mathcal{Z}(\tau)\phi\|_{\bar{\mathcal{H}}} < \epsilon \quad (2.7)$$

for $\tau > \tau_{\phi}$. It can be shown that $\mathcal{Z}(\tau)$ is just the unitary evolution $U(\tau)$ projected into the subspace \mathcal{K} . Since the states which lie in the subspaces \mathcal{D}_{\pm} , in the case of scattering, describe the incoming and outgoing waves which are not influenced by the interaction, the states which lie in \mathcal{K} describe the unstable states, i.e., resonances of the scattering. From this point of view, the Lax-Phillips semigroup is analogous to the reduced motion discussed in the previous section.

This theory constitutes a constructive realization of the theorem of Foias and Nagy [16], which states that given a semigroup on a Hilbert space \mathcal{H} , there is a bigger Hilbert space $\bar{\mathcal{H}}$ which contains it, in which the evolution is a one-parameter unitary group, and this unitary group restricted to \mathcal{H} is that semigroup.*

* The theorem states that this construction is minimal. We conjecture but have not proved that the Lax-Phillips construction is minimal. We thank G. Emch for a discussion of this point.

Flesia and Piron [13] have shown that the quantum theory may be embedded in a Lax-Phillips theory by considering the family of Hilbert spaces of the usual quantum theory on the parameter t as the auxiliary spaces of Lax and Phillips; the large Hilbert space $\bar{\mathcal{H}}$ is then the direct integral of these quantum mechanical spaces over all values of the time t with Lebesgue measure. The form of the theory adopted by Flesia and Piron [13] distinguishes the elements of these equivalence classes, and constructs an S -matrix which maps the auxiliary space in the incoming translation representation to the auxiliary space of the outgoing one. In the model that they use to illustrate this structure, this map corresponds to a pre-asymptotic form of the S -matrix of the usual scattering theory. Their model assumes that the subspaces $\mathcal{D}_+, \mathcal{D}_-$ are represented in the “free” representation, for which the free evolution is translation, by $L^2(-\infty, \rho_-; \mathcal{H})$, $L^2(\rho_+, \infty; \mathcal{H})$, respectively. In the limit in which the interval between the two semi-infinite regions of support tends to infinity, their S -matrix becomes the usual S -matrix. In this construction, Flesia and Piron assume the form

$$\psi_{t+\tau}^\tau = W_t(\tau)\psi_t^0, \quad (2.8)$$

where, since $W_t(\tau)$ represents an evolution, it follows that

$$W_{t+\tau_1}(\tau_2)W_t(\tau_1) = W_t(\tau_1 + \tau_2). \quad (2.9)$$

Lax and Phillips prove that the S -matrix (in their construction) is a multiplicative operator in the spectral representation of the generator of the unitary evolution K (which is the Fourier transform of the translation representation), i.e.,

$$(S\psi)_\sigma = S(\sigma)\psi_\sigma,$$

and that the eigenvalues of the generator of the semigroup $\mathcal{Z}(\tau)$ correspond to the singularities of the analytic continuation of $S(\sigma)$. The eigenstates corresponding to these eigenvalues are analogous to the generalized eigenstates found in the framework of Weisskopf and Wigner, as discussed in Section 1. Thus, the S -matrix contains all the information about the unstable states. It can be seen [10], however, that the S matrix obtained from a model in which the evolution is given in the form (2.8) has no t -dependence, and hence its spectral representation is trivial. In this form, one therefore has no relation between the singularities of the S -matrix and the spectrum of the generator of the semigroup.

Although the generalization of Lax-Phillips theory by Flesia and Piron [13] provides a new point of view for scattering theory, we see that to extend the theory further to include a description of the evolution of an unstable system, it is necessary to generalize the law of evolution to that of a nontrivial integral operator over the time.

The most general linear evolution law has the form

$$(U(\tau)\psi)_{t+\tau} = \int_{-\infty}^{+\infty} W_{t,t'}(\tau)\psi_{t'} dt'. \quad (2.10)$$

We shall show that this type of evolution, which goes beyond the formulation of Flesia and Piron [13] and Floquet theory [12], can correspond to unitary evolution in $\bar{\mathcal{H}}$ with a nontrivial S -matrix for which the singularities of its Fourier transform are associated with

the spectrum of the generator of the Lax-Phillips semigroup. As we shall show below, the form of the evolution law (2.10) has a natural realization in Liouville space as well as in the framework of relativistic quantum theory.

Let us now study for this general evolution, some properties of the S -matrix

$$(S\psi)_t = \int S_{t,t'} \psi_{t'} dt',$$

and show that in this general case the S -matrix must have the form $S_{t,t'} = S(t - t')$. Using the definition $S = W_+^{-1} W_-$, where

$$W_{\pm} = s - \lim_{\tau \rightarrow \pm\infty} U(-\tau) U_0(\tau),$$

we find

$$(S\psi)_t = s - \lim_{\tau_1, \tau_2 \rightarrow \infty} (U_0(-\tau_1) U(\tau_1) U(\tau_2) U_0(-\tau_2) \psi)_t$$

But,

$$\begin{aligned} (U_0(-\tau_1) U(\tau_1 + \tau_2) U_0(-\tau_2) \psi)_t &= (U(\tau_1 + \tau_2) U_0(-\tau_2) \psi)_{t+\tau_1} = \\ &= \int W_{t+\tau_1, t'}(\tau_1 + \tau_2) (U_0(-\tau_2) \psi)_{t'} dt' = \\ &= \int W_{t+\tau_1, t'}(\tau_1 + \tau_2) \psi_{t'+\tau_2} dt' = \int W_{t+\tau_1, t'-\tau_2}(\tau_1 + \tau_2) \psi_{t'} dt', \end{aligned}$$

and therefore the matrix elements of S are

$$\begin{aligned} S_{t,t'} &= s - \lim_{\tau_1, \tau_2 \rightarrow \infty} W_{t+\tau_1, t'-\tau_2}(\tau_1 + \tau_2) = \\ &= s - \lim_{\tau'_1, \tau'_2 \rightarrow \infty} W_{t-t'+\tau'_1, -\tau'_2}(\tau'_1 + \tau'_2) = S(t - t') \end{aligned} \quad (2.11)$$

(where $\tau'_1 = \tau_1 + t'$ $\tau'_2 = \tau_2 - t'$) . This is a very important property of the S -matrix, according to which, when one goes to the spectral representation $\hat{\psi}_\sigma = \int e^{-i\sigma t} \psi_t dt$, the S -matrix takes the simple form

$$\hat{S}_{\sigma, \sigma'} = \frac{1}{2\pi} \int e^{-i\sigma t} S_{t,t'} e^{i\sigma' t'} dt dt' = \delta(\sigma - \sigma') \hat{S}(\sigma)$$

where

$$\hat{S}(\sigma) = \int e^{-i\sigma t} S(t) dt, \quad (2.12)$$

i.e., in this basis the S -matrix is diagonal, and the S -operator is multiplication on the subspaces, labeled by σ , of $\{\mathcal{H}_\sigma\}$, the set of (isomorphic) Hilbert spaces which are the Fourier dual to the set $\{\mathcal{H}_t\}$. This result can be obtained also by looking at the definition of the S -matrix,

$$S = s - \lim_{\tau_1, \tau_2 \rightarrow \infty} U_0(-\tau_1) U(\tau_1 + \tau_2) U_0(-\tau_2)$$

from which it follows that

$$SU_0(\tau) = U_0(\tau)S.$$

Since $U_0(\tau)$ is the translation operator one obtains the result $[S, i\partial_t] = 0$ (which correspond to the usual result of scattering theory $[S, H_0] = 0$). It follows from this commutation relation that $S_{t,t'} = S(t - t')$.

We show now that under the general evolution (2.10), the semigroup is contractive. Let us calculate the generator of the semigroup B of $\mathcal{Z}(\tau) = P_+U(\tau)P_-$. We use the free translation representation in which both \mathcal{D}_\pm have definite support properties. In this representation,

$$\mathcal{Z}(\tau) = P_+U(\tau)P_- = E(\rho)U(\tau)(I - E(0)), \quad (2.13)$$

where $E(t)$ is the spectral resolution corresponding to T_0 , the free-time-operator (the conjugate of K_0 which is, in the free translation representation, $-i\partial_t$). Then, the generator (in the subspace \mathcal{K}) of $\mathcal{Z}(\tau)$ is

$$\begin{aligned} B &= i \lim_{\tau \rightarrow 0} \frac{\mathcal{Z}(\tau) - I_{\mathcal{K}}}{\tau} = i \lim_{\tau \rightarrow 0} \frac{E(\rho)(I - iK\tau)(I - E(0)) - I_{\mathcal{K}}}{\tau} = \\ &= E(\rho)K(I - E(0)) = P_+KP_- . \end{aligned} \quad (2.14)$$

According to the requirements on \mathcal{D}_\pm the matrix elements of κ , the self-adjoint kernel over t, t' contained in K distinct from the t -derivative [10], between states from \mathcal{D}_- to \mathcal{D}_+ , or \mathcal{D}_\pm to \mathcal{K} vanish, and therefore

$$B = P_+K_0P_- + \kappa_{\mathcal{K}}. \quad (2.15)$$

An operator B is called dissipative [17][18] if

$$-i((\phi, B\phi) - (B\phi, \phi)) \leq 0, \quad (2.16)$$

for all ϕ in the domain of B . Since $\kappa_{\mathcal{K}}$ is self-adjoint only the first term determines whether the operator is dissipative, i.e., this property does not depend on the perturbation. As shown by Horwitz and Piron[14], the operator $P_+K_0P_-$ is, in fact, dissipative. It is known [18] that $\mathcal{Z}(\tau)$ is a contractive semigroup if and only if its generator is dissipative. It therefore follows, independently of (self-adjoint) interaction, that the semigroup $\mathcal{Z}(\tau)$ is contractive. We see from this [14] the *essential mechanism of Lax-Phillips theory*. The non-self-adjointness of $P_+K_0P_-$ corresponds to the restriction of $-i\partial_t$ to a finite interval, so that, in fact the operator has imaginary eigenvalues. In the presence of interaction (non-trivial κ), these eigenvalues emerge as the actual eigenvalues of B , corresponding to the singularities of $S(\sigma)$.

We remark that the direct integral space provides a framework as a functional space for quantum mechanics in which the Nagy-Foias construction can be realized, i.e., for which unitary evolution can be restricted to a contractive semigroup. We shall now introduce an extension of the conceptual framework which considers the set $\{\psi_t\}$, corresponding to the Lax-Phillips vector ψ , as an ensemble of the same type, for example, as $\{\psi(x)\} \in \mathcal{H}$, where

x is a point of the spectrum of the position observable, in the usual form of the quantum theory. In concluding this section, we investigate some consequences of this interpretation.

In particular, we discuss some properties of the time operator and the realization of the superselection rule in time. In the next section, we discuss the possibility of decoherence in \mathcal{H} induced by the unitary evolution in $\bar{\mathcal{H}}$.

There are three distinct types of time operator. One, which we call the incoming time operator T^{in} , provides a spectral family in terms of which the incoming representation can be constructed, and in which functions in \mathcal{D}_- have definite support and functions in $\bar{\mathcal{H}}$ evolve by translation. In this representation, the norm of the evolving states in $(-\infty, 0)$ must decrease. After sufficient laboratory time τ passes, the states evolve to \mathcal{D}_+ , and in the outgoing representation, provided by the spectral family of the outgoing time operator T^{out} , they have definite support in (ρ, ∞) . The mapping of functions in the incoming representation to the outgoing representation is provided by the Lax-Phillips S -matrix, and the time operators are related by

$$T^{out} = S T^{in} S^\dagger. \quad (2.17)$$

The third type of time operator corresponds to the “free” representation and is related to T^{in}, T^{out} by the Lax-Phillips wave operators. The spectral family for this operator provides the “standard” representation (analogous to Dirac’s choice of “standard” spectral families), which we have used above.

There is an interval, in general, when the system is in interaction, and its state is neither in \mathcal{D}_- nor \mathcal{D}_+ . The expectation value of the operator T^{in} in the state ψ^τ projected into $\mathcal{K} \oplus \mathcal{D}_+$ (corresponding to the projection P_-) can be interpreted as the interaction interval. If the system in interaction is considered as an unstable particle (a resonance), this interval is its *age* after creation at $t = 0$. The expectation value of T^{in} then moves out of $(-\infty, 0)$. The expectation value of T^{in} in the state $P_- \psi^\tau$ is

$$\langle T^{in} \rangle_\tau = \int t | {}_{in} \langle t | P_- \psi^\tau \rangle |^2 dt; \quad (2.18)$$

here, $| {}_{in} \langle t | P_- \psi^\tau \rangle |^2$ is the probability density for the age t at time τ , an intrinsic dynamical property of the system. The positive value that the expectation value develops corresponds to the average age. One can similarly compute the expected time after decay, the expected lifetime, and the expected value of any other observable of interest as a property of the unstable system.

We then understand the subspace \mathcal{K} as corresponding to the unstable system.

The structure of the theory is somewhat similar to the Wigner-Weisskopf idea, in that a subspace is associated with the decaying system. The decay of the system is associated with the probability flow out of the subspace. As in the original Wigner-Weisskopf formulation, the process of decay may be represented as a continuous evolution from the original unstable state to the final state through a changing linear superposition. In this framework, let us choose a vector ψ in the subspace \mathcal{K} to represent the state of an unstable system. Then, under the full evolution,

$$\begin{aligned} (\psi, U(\tau)\psi) &= (\psi, P_{\mathcal{K}} U(\tau) P_{\mathcal{K}} \psi) \\ &= (\psi, \mathcal{Z}(\tau)\psi), \end{aligned} \quad (2.19)$$

so that *the reduced evolution is an exact semigroup*.

Moreover, in the Lax-Phillips theory the expectation value of an observable which is decomposable in the free or outgoing representations, where \mathcal{D}_+ has definite support properties, necessarily reduces to the sum of the expectation values in the subspaces $\mathcal{K} \oplus \mathcal{D}_-$ and in the subspace \mathcal{D}_+ (the decay products), i.e.,

$$\langle A \rangle = \int dt (\psi_t, A\psi_t) = \sum_{M=D_{\pm}, \mathcal{K}} \int dt (\psi_t^M, A\psi_t^M). \quad (2.20)$$

Note that there are no cross terms. *There is, therefore, an exact superselection rule for measurements of the system by means of such decomposable operators.*

3. APPLICATIONS

a. Measurement according to Namiki and Machida

Recently, Machida and Namiki [19] have proposed a measurement theory based on a direct integral space of continuously many Hilbert spaces and a continuous superselection rule. As pointed out by Tasaki *et al* [20], although they had some success, their theory has a conceptual difficulty. Indeed, in their theory, while the apparatus is described by many Hilbert spaces, the system corresponds to a single Hilbert space as in the conventional theory. Thus, one needs to specify the boundary between the system and the apparatus. As discussed by von Neumann, this is impossible.

Most measurement processes are concerned with measurements of observables which are time-independent in the Schrödinger picture. Therefore, if two different Lax-Phillips states give the same expectation value for all time-independent observables, these two states are essentially indistinguishable. In this sense, we define the following:

1. A Lax-Phillips vector $\psi \in \bar{\mathcal{H}}$ is called “effectively pure” if there exists a pure state

$$\rho_0 = \phi_0 \phi_0^*, \quad \phi_0 \in \mathcal{H},$$

such that

$$\langle \hat{A} \rangle_\psi = \text{Tr} \rho_0 A = (\phi_0, A\phi_0), \quad (3.1)$$

where \hat{A} is the “lift” of A on \mathcal{H} to $\bar{\mathcal{H}}$, for every element of the algebra of bounded linear operators associated with the spectral families of the time-independent observables* on the original space \mathcal{H} .

2. A Lax-Phillips vector is called “effectively mixed” if no such (pure) ρ_0 exists.

It can be shown[10][20] that $\psi = \{\psi_t\} \in \bar{\mathcal{H}}$ is effectively pure if and only if it has the form

$$\psi_t = f(t)\phi_0. \quad (3.2)$$

* We wish to emphasize that what is meant is *explicit* time-dependence in the Schrödinger picture; we do not refer here to the dynamical time-dependence that may arise in the Heisenberg picture if A is not a constant of the motion.

We now discuss the possibility of decoherence, or the evolution from effectively pure to effectively mixed states. First, we consider the Schrödinger evolution for a time-dependent Hamiltonian. The solution of the time-dependent Schrödinger equation can always be written formally as $\psi_t = U(t, t')\psi_{t'}$, where $U(t, t')$ satisfies the chain property $U(t, t')U(t', t'') = U(t, t'')$, and can be expressed in terms of the integral of a time-ordered product. We define $W_t(\tau) = U(t + \tau, t)$, and lift the evolution to $\tilde{\mathcal{H}}$ as follows

$$\psi_{t+\tau}^\tau = W_t(\tau)\psi_t, \quad (3.2)$$

where $W_t(\tau)$ is given by (T implies the time-ordered product)

$$W_t(\tau) = T \left(e^{-i \int_t^{t+\tau} H(t') dt'} \right). \quad (3.3)$$

For this kind of time-evolution we obtain

$$\langle \hat{A} \rangle_\psi = \int dt (W_t(\tau)\psi_t, AW_t(\tau)\psi_t)_{\mathcal{H}}, \quad (3.4)$$

where we have taken the normalization as unity. For the effectively pure states we have

$$\langle \hat{A} \rangle_\psi = \int dt |f(t)|^2 (W_t(\tau)\phi_0, AW_t(\tau)\phi_0)_{\mathcal{H}}. \quad (3.5)$$

It follows from our previous argument that the effective state corresponding to (3.5) is mixed-like if $W_t(\tau)\phi_0 \neq W_{t'}(\tau)\phi_0$ (i.e., the state ρ_ψ induced from $\psi_{t+\tau}^\tau = W_t(\tau)\psi_t = f(t)W_t(\tau)\phi_0$ is not pure in \mathcal{H}). This result is true for the generalized evolution $W_{tt'}$ of (2.10) as well.

b. Intrinsic decoherence in classical and quantum Liouville evolution.

It has long been emphasized by Prigogine and his co-workers [21] that the natural description for the evolution of a system with many degrees of freedom is that of the evolution of the density matrix ρ , through the Liouville equation,

$$i \frac{d\rho}{dt} = [H, \rho]. \quad (3.6)$$

The density matrix ρ ($\rho \geq 0, \text{Tr} \rho = 1$) has the property that $\text{Tr} \rho^2 \leq 1$, where the equality is attained only for a pure state. In general, one considers the space of Hilbert-Schmidt operators A for which

$$\text{Tr} A^* A < \infty; \quad (3.7)$$

the positive (normalized) elements of such a space correspond to the physical states, the density matrices. On this space, the commutator with the Hamiltonian H defines a linear operator \mathcal{L} , called the Liouvillian, for which

$$i \frac{d\rho}{d\tau} = \mathcal{L}\rho, \quad (3.8)$$

where one assumes that \mathcal{L} is self-adjoint in the Liouville space.

The Hamiltonian evolution of states in classical mechanics is known by the Liouville theorem to be non-mixing, i.e., to preserve the entropy of the system [22]. The same property holds for the quantum evolution as well, and follows from the unitarity of the evolution operator. This has been an obstacle to the consistent description of irreversible processes from first principles [23]. The usual use of techniques of coarse graining or truncation to achieve a realization of the second law does not follow from basic dynamical laws, and is fundamentally not consistent with the underlying Hamiltonian structure [24]. We shall now show that the existence of a time operator in the Liouville space provides a natural and consistent mechanism for the decoherence of physical states, i.e., that pure states become mixed during the evolution, both for quantum and classical systems.

In particular, for a Hamiltonian of the form of the sum of an unperturbed operator H_0 and a perturbation V , i.e., $H = H_0 + V$, the corresponding Liouvillian is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I. \quad (3.9)$$

Now suppose we consider the “time operator” T_0 , conjugate to \mathcal{L}_0 (with spectrum $(-\infty, \infty)$); it satisfies

$$[T_0, \mathcal{L}_0] = i.$$

Then, in the spectral representation of T_0 ,

$${}_0\langle t|[T_0, \mathcal{L}_0]|t'\rangle_0 = i\delta(t - t'),$$

or

$$(t - t') {}_0\langle t|\mathcal{L}_0|t'\rangle_0 = i\delta(t - t'). \quad (3.10)$$

It follows that

$${}_0\langle t|\mathcal{L}_0|t'\rangle_0 = -i\partial_t\delta(t - t'). \quad (3.11)$$

Hence,

$${}_0\langle t|\mathcal{L}|t'\rangle_0 = -i\partial_t\delta(t - t') + {}_0\langle t|\mathcal{L}_I|t'\rangle_0, \quad (3.12)$$

where the last term is, in general, not diagonal.

The method that we have described above applies as well to the formulation of classical mechanics on a Hilbert space defined on the manifold of phase space which was introduced by Koopman [25] and used extensively in statistical mechanics [24]. Misra [26] has shown that dynamical systems which admit a Lyapunov operator necessarily have absolutely continuous spectrum in $(-\infty, \infty)$; therefore one can construct a time operator on the classical Liouville space for such systems. The expectation value of a t -independent operator defines a reduced density function in the form

$$\int dt \rho_t(\beta),$$

where β is the set of variables remaining after extracting t as a function on the manifold of the measure (phase) space. Since a pure state is defined by a density function concentrated

at a point of the phase space, a state which is effectively pure must have the form $\delta(\beta - \beta_0)$. The equivalence class associated with this reduced density contains mixed states as well, such as $\rho(t, \beta) = \delta(\beta - \beta_0)f(t)$ corresponding to a non-localized function on the phase space. The structure of the theory, and the conclusions we have reached, are therefore identical to those of the quantum case.

c. Relativistic quantum mechanics.

The form of relativistic quantum mechanics introduced by Stueckelberg [27], extended to the many-body case by Horwitz and Piron [28], covariantly describes the evolution of a system according to the Stueckelberg-Schrödinger equation

$$i \frac{\partial \psi_\tau}{\partial \tau} = \frac{p_\mu p^\mu}{2M} \psi_\tau \equiv K_0 \psi_\tau, \quad (3.13)$$

where M is an intrinsic property of the particle (“on-shell” mass). The classical form of this theory has for its Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial K_0}{\partial p_\mu} = \frac{p^\mu}{M}, \quad (3.14)$$

and, therefore, eliminating $d\tau$, one obtains the standard relativistic relation

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{E}. \quad (3.15)$$

Since the d’Alembertian, corresponding to the operator K_0 , has spectrum $(-\infty, \infty)$, there exists an operator ξ which satisfies

$$[K_0, \xi] = i. \quad (3.16)$$

Note that the operator t of the relativistic theory will not serve this purpose, since its commutator with K_0 is iE/M , which only approaches i in the non-relativistic limit.

If ξ is a function of \mathbf{x}, t , we may construct the transformation function $\langle \xi', \beta | x \rangle$ using the defining commutation relation, i.e.,

$$\langle \xi', \beta | K_0 \xi - \xi K_0 | x \rangle = i \langle \xi', \beta | x \rangle,$$

or

$$i \frac{\partial}{\partial \xi'} (\xi' \langle \xi', \beta | x \rangle) - \xi' \left(-\frac{\partial^\mu \partial_\mu}{2M} \right) \langle \xi', \beta | x \rangle = i \langle \xi', \beta | x \rangle,$$

so that we obtain the defining equation [29]

$$i \frac{\partial}{\partial \xi'} \langle \xi', \beta | x \rangle = -\frac{\partial^\mu \partial_\mu}{2M} \langle \xi', \beta | x \rangle. \quad (3.17)$$

We thus see that the relativistic quantum theory provides a natural framework for the Lax-Phillips formulation of the description of an unstable system. It is interesting

that the continuous spectrum of K_0 is essential to the construction; this implies that we must have both positive and negative mass-squared states in the spectrum, i.e., that the so-called tachyons, at least in the form of intermediate states, play a fundamental role in the relativistic description of unstable systems.

One might ask how such a mechanism could survive in the non-relativistic limit. It is interesting to study this limit; even though the Galilean world is an idealization which is not realized physically, the velocity of light c is very large. To study this limit, we consider the condition

$$E - Mc^2 = \varepsilon < \infty, \quad (3.18)$$

for $c \rightarrow \infty$, used, for example, in ref. [30]. We then define the variable m such that

$$E = c\sqrt{p^2 + m^2c^2}, \quad (3.19)$$

so that

$$E - Mc^2 = (m - M)c^2 + mc^2\left\{\sqrt{1 + \frac{p^2}{m^2c^2}} - 1\right\}. \quad (3.20)$$

Defining

$$\eta = (m - M)c^2 \neq 0,$$

we see that

$$E = Mc^2 + \eta + \frac{p^2}{2M} + f(\eta, p^2), \quad (3.21)$$

an integral kernel on the wave functions, where the integral operator is $O(1/c^2)$.

The general structure of the relativistic Lax-Phillips theory therefore remains in the Galilean limit (for finite but large c). The experimental signature of such an additional term in the Hamiltonian would be, for example, an interference effect in time. The experiment would be of the same design as the test of such an effect in the full covariant relativistic theory [31].

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